RESULTANT POLYTOPE f-VECTORS IN FOUR AND FIVE DIMENSIONS

DANIEL PROFILI

Abstract. For a system of polynomials with $A = (A_1, \ldots, A_k)$ as supports, the Newton polytope of the resultant, or resultant polytope, is the convex hull of the resultant monomial exponent vectors in $\mathbb{Z}^n$ and encodes certain combinatorial properties of the resultant polynomial. Using tropical hypersurface fan traversals, we investigate the f-vectors (vectors of face cardinalities) of resultant polytopes in four and five dimensions. Using the software Gfan to perform tropicalization calculations, our experiments support the currently conjectured maximal f-vector $(22, 66, 66, 22)$ for the 4-dimensional case after sampling 200,000 random point configurations with coordinates in the $[0, 10]$ range. For the 5-dimensional case, we sample over 160,000 resultant polytopes and offer an experimental lower bound for the maximal f-vector of $(58, 232, 330, 201, 47)$. Finally, we present some ideas for further computational and theoretical approaches to f-vector characterization using tropical geometry.

Contents

1. Introduction

Algebraic geometry is the mathematical field lying at the intersection of abstract algebra and geometry, employing the methods of group, ring, and field theory to the study of geometric objects and vice versa. The field is typically concerned with finding the zeros of multivariate polynomials and has been studied since antiquity. The major breakthroughs in modern algebraic geometry have come more recently, with many foundational results in the field having been developed since the early 20th century. Hilbert's Nullstellensatz makes explicit the connection between polynomial rings over an algebraically closed field k and zero sets of polynomial systems by formulating a bijection between algebraic varieties in k^n and radical ideals of $k[x_1, \dots, x_n]$. Another foundational theorem was formulated by Bernstein and Kushnirenko in 1975 and states that the number of complex solutions to a system of Laurent polynomials $f_1 = \cdots = f_n = 0$ in the complex algebraic torus $(\mathbb{C}^*)^n$ equals the mixed volume of the polyhedral complex composed of the Newton polytopes of $f_1, \dots, f_n$ [1]. Tropical geometry is a subfield of algebraic geometry which arose in the 1990s and investigates the tropical semiring $(\mathbb{R}, \min, +)$, which has applications in algebra as a framework for linearizing complex nonlinear polynomials.

For a system of polynomials $\mathcal{F} = \{f_1, \dots, f_k\}$ with indeterminate coefficients in $(\mathbb{C}^*)^n$, the *resultant variety* is the algebraic variety (i.e. a set on which some polynomial ideal vanishes) of tuples of coefficient values such that the polynomials in the system $\mathcal{F}$ have a common zero. The *(sparse mixed) resultant polynomial* is the unique polynomial which vanishes on the resultant variety. Finally, the *Newton polytope of the resultant polynomial* (or *resultant polytope*), which provides the geometric connection, is defined as the d-dimensional convex hull in $\mathbb{R}^d$ of the monomial exponent vectors of R.

The goal of this project is to investigate computational methods for characterizing the Newton polytope of the resultant; in particular, we will focus on properties of the resultant polytope f-vector in four and five dimensions. In [2], resultant polytopes through degree 3 were fully characterized, but there does not yet exist a theoretical characterization for polytopes of dimension 4 and higher, so a useful starting point is to employ computational methods. Traditional methods for computing the resultant rely largely on elimination theory and Gröbner bases [13], which are computationally inefficient for many variables or polynomials of high degree. Recent works by [3] and [4] have described methods involving the tropicalization of the polynomials f_i as potentially more efficient avenues for calculating the resultant. Work in [3] has conjectured that the maximum number of vertices for a 4-dimensional resultant polytope is 22, but this is an open problem; thus, this project will use software packages including SageMath and Gfan to attempt to verify this upper bound. We hypothesize that this upper bound is correct, as the methods in [3] are sound and are backed with algebraic-geometric theory, but a computational approach will help to solidify this upper bound where pure theory is lacking. Working toward a characterization of 4- and 5-dimensional resultant polytopes may eventually lead to a characterization of resultant polytopes in all dimensions, which would be a major breakthrough in solving systems of polynomials and in the general field of algebraic geometry.

The question of concavity and log-concavity of f-vectors of general polytopes has been investigated before. The known status of these properties for general polytopes was described by [7], but concavity and log-concavity of resultant polytope f-vectors in particular has not, to our knowledge, been discussed in literature yet. Thus, during our experiments we will also investigate the concavity and log-concavity of f-vectors of 5-dimensional resultant polytopes.

2. Background and literature review

2.1. Algebraic geometry foundations. The application of geometric methods to solving systems of polynomial equations has been a major area of study for algebraic geometers since Bernstein, in 1975, proved the following fundamental result for sparse systems of Laurent polynomials [1].

Throughout, we denote monomials by $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}$ for $a_i \in \mathbf{a}$. Let $A_1, \dots, A_k$ be a family of fixed finite subsets of $\mathbb{Z}^n$ and let $\mathcal{F} = \{f_i(x) = \sum_{\mathbf{a} \in A_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}\}$ be a collection of polynomials with the A_i as supports, and let $Q_i = \operatorname{conv}(A_i)$ be the convex hulls of the support vectors (called the *Newton polytope* of f_i). Our primary space of interest is $(\mathbb{C}^*)^n = \{\mathbb{C} \setminus 0\}^n$, the *$n$-dimensional complex torus.*

Theorem 2.1 (Bernstein). *For almost all choices of coefficients $c_{i,\mathbf{a}} \in \mathbb{C}^*$, the number of common zeros of $\mathcal{F}$ in the complex torus $(\mathbb{C}^*)^n$ equals the mixed volume of $Q_1, \dots, Q_k$.*

The *mixed volume* is defined as the coefficient of $\lambda_1\lambda_2 \dots \lambda_k$ in the polynomial

$$R(\lambda_1, \dots, \lambda_k) = \mathrm{vol}(\lambda_1 Q_1 + \lambda_2 Q_2 + \cdots + \lambda_k Q_k),$$

where here the addition of polytopes denotes the *Minkowski sum.* In short, Theorem 2.1 provides a connection between the algebraic properties of $\mathcal{F}$ and a geometric property of a certain set of polytopes.

This area of study is closely linked to the study of convex polytopes, whose seminal text was published around the same time [5] and who had widely accessible texts published soon after, such as the book by Brøndsted [6]. The connection between the algebraic properties of polynomial systems and convex polytopes was further solidified in the 1990s with algorithmic approaches found in [1] and [2], whose algorithms form the computational basis for the present study.

2.2. **f-vectors.** For a d-dimensional polytope P, the f-vector of P [5] is the ordered list of positive integers

$$(f_{-1}, f_0, f_1, \dots, f_{d-1}, f_d)$$

where f_i equals the number of faces of P having dimension i. For the most part, in this study we focus on maximal values for f_0, which is the number of vertices. Trivially, for every polytope we have $f_{-1} = f_d = 1$, being the empty face and the full polytope, respectively, so we typically omit these and simply write $(f_0, \dots, f_{d-1})$.

The f-vectors for certain classes of polytopes encode important combinatorial properties; for example, f-vectors of simplices (i.e. n-dimensional analogues of triangles) correspond to rows from Pascal's triangle. Further, there are certain restrictions on the structure of f-vectors for general polytopes with regard to properties of integer sequences which can be applied to f-vectors, being

- concavity: $2f_k \geq f_{k-1} + f_{k+1}$ for all $k = 1, \ldots, d-2$
- log-concavity: $f_k^2 \geq f_{k-1}f_{k+1}$ for all $k = 1, \ldots, d-2$
- unimodality: there exists $k \in \{0, \ldots, d-1\}$ such that $f_0 \leq \cdots \leq f_k \geq \cdots \geq f_{d-1}$.

It is relatively straightforward to see that each property implies the properties below it. In [7] the status of these properties for several low-dimensional classes of polytopes is enumerated, and we summarize these results in Table 1.

Dimension	≤ 4	5	6	7	≥ 8
Concavity	✓	✗	✗	✗	✗
Log-concavity	✓	?	?	?	✗
Unimodality	✓	✓	?	?	✗

Table 1. Known properties of polytope f-vectors, adopted from [7].

As shown in Table 1, the lowest-dimensional property still in question is log-concavity for 5-dimensional polytopes; thus, for the current study, we will focus on the question of log-concavity for 5-dimensional resultant polytopes (considered as a subset class of general 5-dimensional polytopes), as our computational framework makes log-concavity simple and fast to compute for large numbers of resultant polytopes.

2.3. **Tropical geometry.** The subfield of tropical geometry, or the study of the min-plus semiring, came into being in the late 1990s ([8], [9]). With its origins in optimization theory, the connection was soon drawn to solving systems of polynomial equations, and from there many connections were drawn to polytope theory and the theory of resultants [10]. More recently, it has been shown in [4] that the tropicalization of polynomial systems has great potential for the development of efficient algorithms for solving these systems via the use of the resultant. The area of tropical geometry is therefore of great mathematical interest with regards to solving polynomial systems via the resultant.

A common thread in more recent research, including [4], [3], [11], and [12], has been the development of efficient algorithms to implement the theory of convex polytopes and resultants; in particular, [4] demonstrates the use of tropical geometry as a way of projecting nonlinear systems onto tropical spaces which, in some sense, are piecewise linear.

3. Resultants

Let $A = A_1, \ldots, A_k$ be a family of (not necessarily distinct) multisets of integer point configurations in $\mathbb{Z}^n$. Let $\mathcal{F} = \{f_i(x) = \sum_{\mathbf{a} \in A_i} c_{i,\mathbf{a}} \mathbf{x}^{\mathbf{a}}\}$ be the set of polynomials with the A_i as supports having coefficients in $\mathbb{C}^*$. Finally, let Z be the set of tuples of coefficients $c_{i,\mathbf{a}}$ such that the system $f_1 = f_2 = \cdots = f_k = 0$ has a solution in $(\mathbb{C}^*)^n$.

Definition 3.1. The *resultant variety* $\mathcal{R}(A)$ is the (algebraic) closure of Z in $(\mathbb{C}^*)^n$.

The *sparse mixed resultant* R is the unique irreducible polynomial in $\mathbb{Z}[c_{i,\mathbf{a}}]$ which vanishes on $\mathcal{R}(A)$, and is well-defined so long as $\operatorname{codim}(\mathcal{R}(A)) = 1$. Further, in [2] Sturmfels showed that whenever the codimension of the resultant variety is 1, then the sparse mixed resultant coincides with the resultant of the polynomials in $\mathcal{F}$, considered with respect to the lattice spanned by the support vectors A. Thus, unless stated otherwise, for this study we shall consider only (without loss of generality) resultant varieties of codimension 1 so that the resultant polynomial is well-defined.

3.1. Resultant polytopes.

The *Newton polytope of the resultant* $\mathcal{N}(R)$, or resultant polytope for short, is defined as the convex hull of the exponent tuples of the resultant polynomial; specifically:

$$\mathcal{N}(R) = \operatorname{conv}\{\mathbf{a} \mid \mathbf{a} \in \mathbb{Z}^n \wedge \mathbf{x}^{\mathbf{a}} \text{ appears in } R\}.$$

The combinatorial properties of the resultant polytope have been extensively studied and are, in general, much less computationally intensive to calculate than the resultant itself.

When the resultant polytope is a hypersurface (i.e. has codimension 1), the resultant polytope possesses a number of useful combinatorial and tropical-geometric properties. In order to limit ourselves to only the consideration of those such systems, the following result from Sturmfels [2] is helpful.

Theorem 3.2. *Let $m_i = |A_i|$ and $m = \sum_{i=1}^{k} m_i$. Then the dimension of the resultant polytope equals $m - 2n - 1$.*

For our goal of studying 4-dimensional resultant polytopes, we gain a useful generalization from Theorem 6.2 in [2]: to summarize, the only distinct resultant polytopes are those with $m_i \geq 3$. Combining this with Theorem 3.2 and solving the resulting system of equations with $m - 2n - 1 = \dim(\mathcal{N}(R)) = 4$ and $m_i \geq 3$ gives a polynomial system description of $n = 2$ and $m_1 = m_2 = m_3 = 3$; that is, we need only consider polynomial systems in two variables consisting of three equations with three monomial terms each. In the terminology of [3], we call this case $(3, 3, 3)$. In the sufficiently generic case (with no A_i containing repeated points), these systems will also have resultant polytopes of codimension 1. For 5-dimensional resultants, we focus on the $(4, 3, 3)$ case.

3.2. **Computational geometry.** More recently, there has been extensive development into the area of computational algebraic geometry, which uses software and algorithms to efficiently solve problems in algebra and geometry. This thread of research began in the 1960s with the development of Gröbner bases, the calculation of which can be done using Buchberger's algorithm [13] for a set of polynomials in n variables of degree at most d in roughly $O(d^{2^n})$ time. The resultant, which is the central object under consideration in the present study, is traditionally calculated using elimination theory techniques [13] making use of Gröbner bases; however, the method is not computationally efficient thanks to the double-exponential running time of Buchburger's algorithm, so there is need for faster algorithms to calculate resultants. Well-known computer algebra systems such as Macaulay2 and SageMath have the capability to compute resultants via elimination theory techniques, and Gfan [14] can perform Gröbner fan traversals as well as compute various properties of tropical hypersurfaces, including their f-vectors.

In the last few years, [3] and [4] have posed the open problem of characterizing n-dimensional resultant polytopes. In [2], all possible resultant polytopes of dimension up to 3 were enumerated, and [3] conjectured an upper bound for the 4-dimensional resultant polytopes using the f-vector, .

With regards to the 4-dimensional resultant polytopes, there are several interesting conjectures posed by [3] which we will investigate. First:

Conjecture 3.3. *The maximum f-vector of a 4-dimensional resultant polytope is* $(22, 66, 66, 22)$*; that is, for every f-vector (f_0, f_1, f_2, f_3), we have $f_0 \leq 22$ and $f_1 \leq 66$.*

The structure of the maximal f-vectors, being those f-vectors with $f_2 = 66$ and $f_3 = 22$, is also not fully understood. Another question worth investigating is

Conjecture 3.4. *For maximal f-vectors (f_0, f_1, f_2, f_3), we have $f_0 = f_3$. Further, if $f_0 \geq 10$, we have $f_1 \geq f_2$.*

The current study will therefore attempt to continue the work of [3] in order to verify or improve the upper bound on the magnitude of the resultant polytope f-vector in four dimensions as well as investigate f-vectors for resultant polytopes in five dimensions. These questions are important to the study of resultants because there is currently no known upper bound on the magnitude of the f-vector in the general case of all resultant polytopes. When dealing with an algebraic object as complicated as the resultant, any properties that can be deduced in general about it are very useful to the development of further theory. Knowledge of such an upper bound could also be useful in the development of algorithms and computational methods for calculating the resultant, which in turn can be used to solve systems of polynomial equations.

4. Experimental results

4.1. **Methods.** Our computer experiments were conducted using Sage for polytope visualization and the software package Gfan, developed by Anders Jensen [14], to compute resultant polytopes. To compute f-vectors of the resultant polytope, the tropical hypersurface of the resultant was computed using the Gfan function `gfan_resultantfan` which constructs the normal fan of the resultant polytope via the characterization from Theorem 2.9 from [4]. For our experimental approach, the key fact is that when $\mathcal{TR}(A)$ is a hypersurface (i.e. has codimension 1), its f-vector equals the f-vector of the resultant polytope for the corresponding system having A as its supports.

In light of this, our experiments ran as Python scripts to generate point configurations and computed tropical resultants in parallel using Gfan. In contrast to the experimental method used in [3], which was exhaustive for a certain limited set of configurations, we opted to randomly generate configurations from a much larger range in order to compute resultants from more exotic point configurations. This method was used because the enumerative approach is simply not viable for exploring large ranges of point configurations; replication of the experiment from [3], which was enumeration of all possible $(3,3,3)$ point configurations of the form (A_0, A_1, A_2) where

$$A_0 = \{(0,0),(0,1),(1,0)\}$$

$$A_1 = \{(0,0), a_1, a_2\},$$

$$A_2 = \{(0,0), a_3, a_4\},$$

and $a_i \in \{(j,k) \mid j,k \in \mathbb{N} \wedge j,k \leq 5\}$, took over 10 hours using Python 3.9's `multiprocessing` library on a modern computer with a 16-core processor and 16 GB RAM. Although the parallel processing tropical approach already gives an edge over traditional methods for computing resultants, without high-throughput computing or a greater degree of parallelism the brute-force enumerative approach is infeasible.

4.2. **Resultant polytopes in four dimensions.** In order to gather experimental evidence for the conjectured maximal f-vector of $(22, 66, 66, 22)$ for 4-dimensional resultant polytopes, we generated random integer point configurations of the form (A_1, A_2, A_3), where

$$A_i = \{(a_{1,i}, a_{2,i}), (a_{3,i}, a_{4,i}), (a_{5,i}, a_{6,i})\}$$

and the $a_{i,j} \in [0, 10] \subset \mathbb{Z}$ are uniformly randomly sampled integers. The aggregate set of unique f-vectors achieved via this random sampling approach is given in Table 2.

4.3. **Resultant polytopes in five dimensions.** In our consideration of five-dimensional resultant polytopes, we considered $(4,3,3)$ configurations with coordinates in the $[0,10]$ range, which, by Theorem 3.2, generate resultant polytopes in five dimensions when the point configurations are sufficiently generic (i.e. having no repeated or collinear points).

The following examples demonstrate that some, but not all, f-vectors of five-dimensional resultant polytopes are concave.

(4, 6, 4)	(5, 8, 5)	(6, 11, 7)
(6, 13, 13, 6)	(7, 15, 14, 6)	(8, 20, 20, 8)
(8, 18, 17, 7)	(9, 22, 21, 8)	(9, 20, 18, 7)
(9, 24, 25, 10)	(10, 25, 24, 9)	(10, 24, 23, 9)
(10, 25, 25, 10)	(10, 26, 25, 9)	(11, 29, 29, 11)
(11, 27, 25, 9)	(11, 28, 27, 10)	(11, 29, 28, 10)
(11, 26, 23, 8)	(12, 30, 27, 9)	(12, 30, 28, 10)
(12, 33, 33, 12)	(12, 32, 31, 11)	(12, 29, 26, 9)
(13, 33, 30, 10)	(13, 33, 31, 11)	(13, 34, 32, 11)
(13, 37, 37, 13)	(13, 32, 29, 10)	(13, 34, 33, 12)
(14, 36, 33, 11)	(14, 37, 36, 13)	(14, 40, 40, 14)
(14, 38, 37, 13)	(14, 35, 32, 11)	(14, 38, 36, 12)
(14, 37, 34, 11)	(14, 38, 38, 14)	(14, 37, 35, 12)
(15, 42, 41, 14)	(15, 39, 36, 12)	(15, 41, 39, 13)
(15, 41, 40, 14)	(15, 40, 36, 11)	(15, 40, 37, 12)
(15, 40, 38, 13)	(15, 42, 42, 15)	(16, 45, 43, 14)
(16, 42, 39, 13)	(16, 44, 41, 13)	(16, 45, 44, 15)
(16, 44, 42, 14)	(16, 43, 39, 12)	(16, 43, 41, 14)
(16, 46, 45, 15)	(16, 43, 40, 13)	(16, 46, 46, 16)
(17, 47, 45, 15)	(17, 49, 49, 17)	(17, 47, 43, 13)
(17, 49, 47, 15)	(17, 46, 43, 14)	(17, 49, 48, 16)
(17, 47, 44, 14)	(17, 48, 46, 15)	(17, 48, 47, 16)
(17, 48, 45, 14)	(17, 50, 50, 17)	(18, 52, 51, 17)
(18, 53, 53, 18)	(18, 54, 54, 18)	(18, 51, 48, 15)
(18, 52, 50, 16)	(18, 51, 49, 16)	(18, 53, 51, 16)
(19, 55, 54, 18)	(19, 54, 52, 17)	(19, 55, 51, 15)
(19, 55, 52, 16)	(19, 57, 57, 19)	(19, 56, 54, 17)
(19, 56, 56, 19)	(20, 57, 51, 14)	(20, 60, 60, 20)
(20, 59, 57, 18)	(20, 58, 54, 16)	(21, 62, 60, 19)
(21, 63, 63, 21)	(22, 66, 66, 22)	

TABLE 2. Unique 4d f-vectors from 200,000 generic random configurations.

Example 4.1. Let $A_1 = \{(3,2),(5,2),(0,1),(1,2)\}$, $A_2 = \{(5,1),(3,4),(4,1)\}$, and $A_3 = \{(0,0),(3,2),(4,2)\}$. Then $\mathcal{N}(R)$ is five-dimensional with f-vector

$$(35, 129, 169, 94, 21),$$

which is log-concave but not concave, since

$$2f_3 = 188 \not\geq f_2 + f_4 = 21 + 169 = 190.$$

Example 4.2. Let $A_1 = \{(7,5),(10,2),(10,1),(2,2)\}$, $A_2 = \{(6,0),(2,2),(7,10)\}$, and $A_3 = \{(1,9),(2,10),(8,5)\}$. Then $\mathcal{N}(R)$ is five-dimensional with f-vector

$$(58, 232, 330, 201, 47),$$

which is both log-concave and concave.

Table 3 contains all the unique f-vectors computed after randomly generating over 160,000 point configurations. The set of unique f-vectors for 5-dimensional resultant polytopes is much larger than the corresponding set from the 4-dimensional case; as such, Table 3 contains only the 100 largest (sorted by f_0 value) f-vectors from our random sampling experiments. Interestingly, observe that none of the f-vectors in Table 3 is symmetric, in contrast to f-vectors in the 4-d case. Since the maximal f-vectors in the 3-d [2] and (hypothesized in) the 4-d cases are symmetric, this suggests that our experiments may have not found the maximal 5-d f-vector.

5. Discussion

The primary significance of these findings with regards to 4-d resultant polytopes is the large amount of supporting evidence for open problems 3.3 and 3.4, taken from a much larger sample space than in [3]. For the 5-dimensional case, open problem 6.1 posits an experimental estimate regarding the size of maximal 5-d resultant polytope f-vectors, which have not been investigated thoroughly in literature yet. Further, our experiments support the conjecture that 5-dimensional resultant polytope f-vectors are log-concave.

A major factor limiting the efficacy of this approach is the limitations of the software Gfan used to compute resultants. While it is robust for point configurations relatively small in magnitude, the program begins to operate slowly and unreliably with points having coordinates much higher than 10, thanks to memory limitations and integer overflow. Further, computation time predictably increases with higher-dimension resultant polytopes; during our experiments, computing a single 4-d f-vector took, on average, 0.03 seconds, while computing a single 5-d f-vector took between 0.13 and 3 seconds, depending on the magnitudes of the points in the configuration.

(48, 188, 262, 156, 36)	(48, 189, 259, 149, 33)	(48, 190, 267, 160, 37)
(48, 190, 266, 158, 36)	(48, 187, 252, 143, 32)	(48, 188, 260, 151, 33)
(48, 188, 260, 152, 34)	(48, 181, 230, 120, 25)*	(48, 191, 266, 155, 34)
(48, 188, 255, 144, 31)	(48, 189, 266, 160, 37)	(48, 189, 261, 151, 33)
(48, 185, 255, 150, 34)	(48, 187, 260, 155, 36)	(48, 187, 258, 151, 34)
(48, 186, 248, 138, 30)*	(48, 190, 265, 156, 35)	(48, 187, 259, 153, 35)
(48, 191, 266, 156, 35)	(49, 194, 272, 162, 37)	(49, 192, 267, 158, 36)
(49, 192, 266, 155, 34)	(49, 195, 275, 165, 38)	(49, 193, 270, 161, 37)
(49, 195, 272, 159, 35)	(49, 191, 263, 153, 34)	(49, 194, 271, 160, 36)
(49, 189, 259, 151, 34)	(49, 189, 258, 149, 33)	(49, 194, 269, 157, 35)
(49, 194, 268, 154, 33)	(49, 192, 266, 156, 35)	(49, 192, 263, 150, 32)
(49, 196, 276, 165, 38)	(49, 191, 264, 155, 35)	(49, 193, 267, 155, 34)
(49, 191, 265, 157, 36)	(49, 194, 273, 164, 38)	(49, 194, 268, 155, 34)
(49, 190, 262, 154, 35)	(49, 192, 268, 160, 37)	(49, 195, 272, 160, 36)
(49, 192, 265, 154, 34)	(49, 193, 265, 153, 34)	(50, 193, 256, 141, 30)*
(50, 199, 281, 169, 39)	(50, 188, 238, 123, 25)*	(50, 197, 274, 160, 35)
(50, 196, 273, 162, 37)	(50, 196, 272, 160, 36)	(50, 194, 260, 145, 31)*
(50, 197, 270, 154, 33)	(50, 198, 274, 159, 35)	(50, 198, 275, 161, 36)
(50, 189, 242, 128, 27)*	(50, 197, 271, 156, 34)	(50, 199, 278, 163, 36)
(50, 199, 278, 164, 37)	(50, 196, 267, 152, 33)	(50, 194, 265, 152, 33)
(50, 189, 241, 125, 25)*	(50, 190, 246, 133, 29)*	(50, 195, 271, 161, 37)
(50, 200, 282, 169, 39)	(50, 195, 263, 147, 31)	(50, 195, 270, 159, 36)
(50, 198, 278, 166, 38)	(50, 194, 259, 143, 30)*	(50, 194, 260, 146, 32)
(50, 197, 276, 165, 38)	(50, 197, 271, 157, 35)	(50, 195, 264, 151, 34)
(51, 200, 279, 166, 38)	(51, 202, 280, 163, 36)	(51, 203, 284, 167, 37)
(51, 203, 284, 168, 38)	(51, 201, 276, 158, 34)	(51, 202, 284, 170, 39)
(51, 204, 288, 173, 40)	(51, 203, 287, 173, 40)	(52, 200, 272, 156, 34)
(52, 206, 286, 167, 37)	(52, 208, 294, 177, 41)	(52, 207, 290, 171, 38)
(52, 207, 290, 172, 39)	(52, 205, 288, 173, 40)	(52, 202, 279, 163, 36)
(53, 212, 300, 181, 42)	(53, 210, 292, 171, 38)	(53, 208, 291, 174, 40)
(53, 211, 296, 176, 40)	(53, 209, 288, 166, 36)	(53, 211, 296, 175, 39)
(54, 216, 306, 185, 43)	(54, 215, 302, 179, 40)	(54, 214, 298, 175, 39)
(55, 220, 312, 189, 44)	(56, 224, 318, 193, 45)	(57, 225, 312, 182, 40)
(58, 232, 330, 201, 47)		

TABLE 3. 100 largest 5d f-vectors from ∼160,000 random configurations. Asterisks indicate non-concavity.

6. Conclusion

6.1. **Future directions.** There is ample room to improve the computational approach used in this study. To improve the ability of the brute-force approach, one avenue that could be explored is the use of more computing power. Because configurations, whether generated randomly or enumeratively, can be created and their resultants computed independently of one another, there is opportunity to explore increased parallelism in the computational approach. This could be accomplished via processors with more cores but is likely better approached via high-throughput computing. The fact that computations are done relatively quickly and in parallel means that high-throughput computing systems could perform many more resultant computations much more quickly than any single computer.

There are also theoretical avenues by which these problems can be approached. First of which is the fact that the tropical approach detailed in [4] can be used to derive certain properties of resultant f-vectors. For example, it follows easily from the connection between resultant polytopes and arrangement of tropical hypersurfaces that linear translations of polygons in point configurations produce combinatorially equivalent resultant polytopes; in particular, their resultant polytopes have the same f-vector. Properties like these have the potential to be exploited to derive properties of resultant f-vectors.

6.2. **Open problems.** Due to the lack of a theoretical approach taken in this study, as well as limited availability of computing power, there are still a number of open problems to explore with regard to the properties of resultant polytopes. In addition to Conjectures 3.3 and 3.4, for which we failed to find any counterexamples, we have the following open problems about the structure of 5-dimensional resultant f-vectors.

Open problem 6.1. *For 5-dimensional resultant polytopes, is it true that* $f_0 \leq 58$*?*

Open problem 6.2. *Are f-vectors of 5-dimensional resultant polytopes log-concave? What about 5-dimensional polytopes in general?*

References

[1] Birkett Huber and Bernd Sturmfels. A polyhedral method for solving sparse polynomial systems. *Mathematics of Computation*, 64(212):1541–1555, 1995.

[2] Bernd Sturmfels. On the Newton Polytope of the Resultant. *Journal of Algebraic Combinatorics*, 3:207–236, 1994.

[3] Alicia Dickenstein, Ioannis Z. Emiris, and Vissarion Fisikopoulos. Combinatorics of 4-dimensional Resultant Polytopes. Preprint, 2013.

[4] Anders Jensen and Josephine Yu. Computing tropical resultants. *Journal of Algebra*, 387:287–319, September 2013.

[5] Günter M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1995.

[6] Arne Brondsted. *An Introduction to Convex Polytopes*. Springer-Verlag, New York, 1982.

[7] Axel Werner. Unimodality and convexity of f-vectors of polytopes. Preprint, 2019.

[8] Erwan Brugallé and Kristin Shaw. A bit of tropical geometry. January 2014.

[9] Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald. First steps in tropical geometry. 2005.

[10] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*. American Mathematical Society, Providence, Rhode Island, 2010.

[11] Diane Maclagan and Rekha Thomas. Computational Algebra and Combinatorics of Toric Ideals. 2009.

[12] Bernd Sturmfels. *Solving Systems of Polynomial Equations*. Berkeley, CA, 2002.

[13] David Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms*. Springer, third edition, 2012.

[14] Anders N. Jensen. Gfan, a software system for Gröbner fans and tropical varieties. http://home.imf.au.dk/jensen/software/gfan/gfan.html.

[15] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations*, volume 25 of *Algorithms and Computation in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[16] Jesús A. De Loera, Sonja Petrović, Lily Silverstein, Despina Stasi, and Dane Wilburne. Random monomial ideals. *Journal of Algebra*, 519:440–473, 2019.

[17] Bernd Sturmfels. *Grobner Bases and Convex Polytopes*. American Mathematical Society, 1991.

[18] Rekha R Thomas. *Lectures in Geometric Combinatorics*. 2006.